COMP 2805 — Solutions Assignment 2

Question 1: Give regular expressions describing the following languages. In all cases, the alphabet is $\{0, 1\}$.

- $\{w : w$ contains at least five 1s$\}$.

  Solution:
  $$0^*10^*10^*10^*10^*1(0 \cup 1)^*$$
  or
  $$(0 \cup 1)^*1(0 \cup 1)^*1(0 \cup 1)^*1(0 \cup 1)^*1(0 \cup 1)^*1(0 \cup 1)^*$$

- $\{w : w$ contains at least two 1s and at most one 0$\}$.

  Solution:
  $$111^* \cup 1^*(011 \cup 101 \cup 110)1^*$$

- $\{w : w$ contains an even number of 1s or exactly two 0s$\}$.

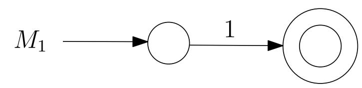
  Solution:
  $$0^* \cup (0^*10^*10^*)^* \cup 1^*01^*01^*$$

Question 2: Use the construction given in class (and described in the notes) to convert the regular expression
$$(((10)^*(00)) \cup 10)^*$$
to an NFA. The alphabet is $\{0, 1\}$.

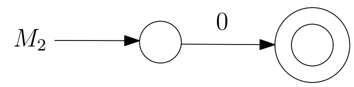
Solution: We first consider how the regular expression is "built":

- Take the regular expressions 1 and 0, and combine them into the regular expression 10.
- Take the regular expression 10, and turn it into the regular expression $(10)^*$.
- Take the regular expressions 0 and 0, and combine them into the regular expression 00.
- Take the regular expressions $(10)^*$ and 00, and combine them into the regular expression $(10)^*00$.
- Take the regular expressions $(10)^*00$ and 10, and combine them into the regular expression $(10)^*00 \cup 10$.
- Take the regular expression $(10)^*00 \cup 10$, and turn it into the regular expression $((10)^*00 \cup 10)^*$.

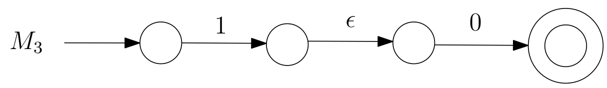
First, we construct an NFA M_1 that accepts the language described by the regular expression 1:



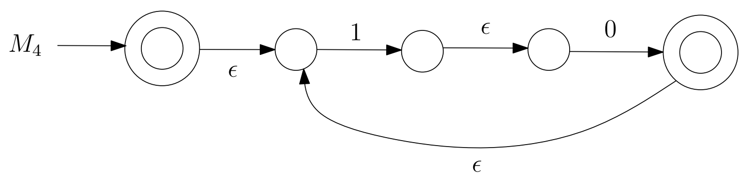
Next, we construct an NFA M_2 that accepts the language described by the regular expression 0:



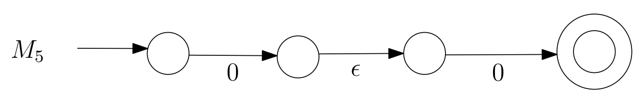
Next, we apply the concatenation construction to M_1 and M_2. This gives an NFA M_3 that accepts the language described by the regular expression 10:



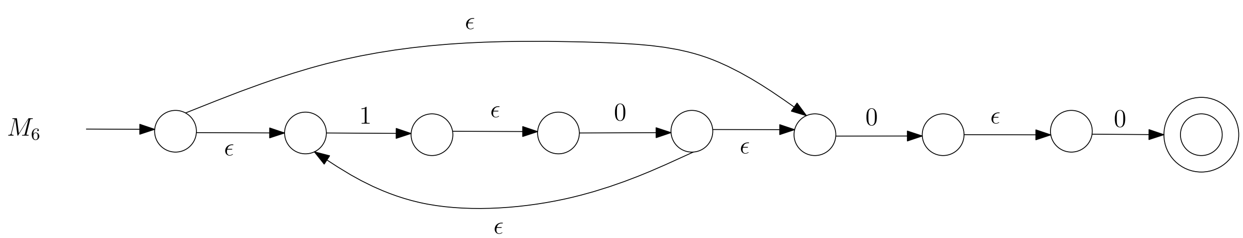
Next, we apply the star construction to M_3. This gives an NFA M_4 that accepts the language described by the regular expression $(10)^*$:



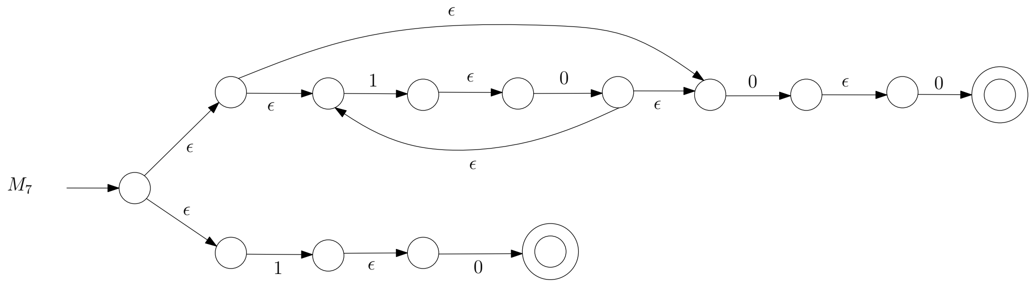
Next, we apply the concatenation construction to M_2 and M_2. This gives an NFA M_5 that accepts the language described by the regular expression 00:



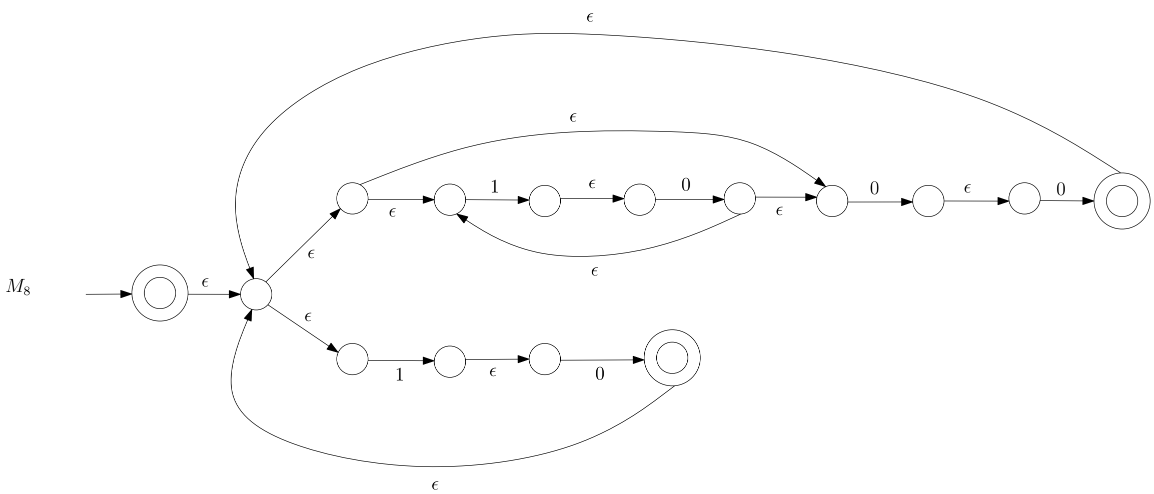
Next, we apply the concatenation construction to M_4 and M_5. This gives an NFA M_6 that accepts the language described by the regular expression $(10)^*00$:



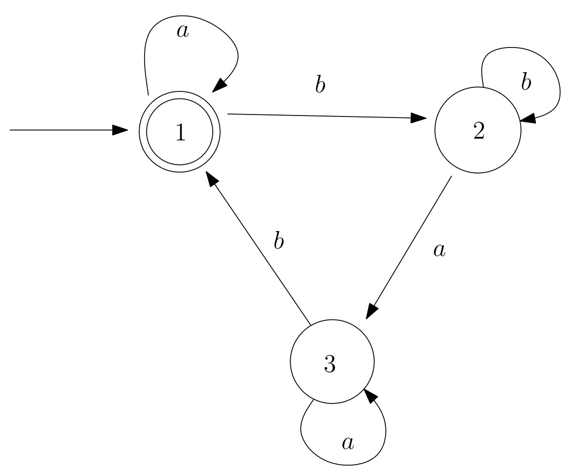
Next, we apply the union construction to M_6 and M_3. This gives an NFA M_7 that accepts the language described by the regular expression $(10)^*00 \cup 10$:



Finally, we apply the star construction to M_7. This gives an NFA M_8 that accepts the language described by the regular expression $((10)^*00 \cup 10)^*$:



Question 3: Use the construction given in class (and described in the notes) to convert the following DFA to a regular expression.



Solution: For each state $i = 1, 2, 3$, we define L_i to be the set of all strings w in $\{a, b\}^*$ such that the path in the state diagram that starts in state i and that corresponds to w ends in the accept state 1. We obtain the following set of equations:

$$L_1 = \epsilon \cup aL_1 \cup bL_2 \quad (1)$$
$$L_2 = aL_3 \cup bL_2 \quad (2)$$
$$L_3 = aL_3 \cup bL_1 \quad (3)$$

Since 1 is the start state, we need a regular expression for L_1.

We use the following tool to solve these equations:

If $L = BL \cup C$ and $\epsilon \notin B$, then $L = B^*C$.

We solve the equations (1), (2), and (3), in the following way: From (3), we obtain

$$L_3 = a^*bL_1,$$

which we substitute into (2), giving

$$L_2 = aa^*bL_1 \cup bL_2,$$

which we rewrite as

$$L_2 = bL_2 \cup aa^*bL_1,$$

which solves to

$$L_2 = b^*aa^*bL_1,$$

which we substitute into (1), giving

$$L_1 = \epsilon \cup aL_1 \cup bb^*aa^*bL_1,$$

which we rewrite as

$$L_1 = (a \cup bb^*aa^*b)\, L_1 \cup \epsilon,$$

which solves to

$$L_1 = (a \cup bb^*aa^*b)^* \,\epsilon = (a \cup bb^*aa^*b)^* \,.$$

Hence, the regular expression describing the language accepted by the DFA is

$$(a \cup bb^*aa^*b)^* \,.$$

By the way,

$$(a \cup bb^*aa^*b)^* \,\epsilon$$

is also correct.

Question 4: Use the pumping lemma to prove that the following languages are not regular.

1. $\{a^nb^mc^{n+m} : n \geq 0, m \geq 0\}$.
2. $\{a^nb^ma^n : n \geq 0, m \geq 0\}$.

3. $\{a^{2^n} : n \geq 0\}$. (Remark: a^{2^n} is the string consisting of 2^n many a's.)

4. $\{a^n b^m c^\ell : n \geq 0, m \geq 0, \ell \geq 0, n^2 + m^2 = \ell^2\}$.

Solution: First, we do

$$A = \{a^n b^m c^{n+m} : n \geq 0, m \geq 0\}.$$

Assume the language A is regular. Let $p \geq 1$ be the pumping length, as given by the pumping lemma. Let $s = a^p b^p c^{2p}$. Then $s \in A$ and $|s| = 4p \geq p$. Hence, by the pumping lemma, we can write $s = xyz$, where

1. $y \neq \epsilon$,

2. $|xy| \leq p$, and

3. $xy^i z \in A$, for all $i \geq 0$.

Since $|xy| \leq p$, the string y contains only as. Since $y \neq \epsilon$, the string y contains at least one a. Hence, the string $xy^2z = xyyz$ contains (i) more than p many as, (ii) exactly p many bs, and (iii) exactly $2p$ many cs. That is, in the string xy^2z, the number of as plus the number of bs is larger than the number of cs. But this means that xy^2z is not in the language A. This is a contradiction, because, by the pumping lemma, this string is an element of A. So we have a contradiction, and we can conclude that A is not regular.

Next, we do

$$B = \{a^n b^m a^n : n \geq 0, m \geq 0\}.$$

Assume the language B is regular. Let $p \geq 1$ be the pumping length, as given by the pumping lemma. Let $s = a^p b a^p$. Then $s \in B$ and $|s| = 2p + 1 \geq p$. Hence, by the pumping lemma, we can write $s = xyz$, where

1. $y \neq \epsilon$,

2. $|xy| \leq p$, and

3. $xy^i z \in B$, for all $i \geq 0$.

Since $|xy| \leq p$, the string y is contained in the leftmost a-block of the string s. We also know that y is non-empty. Consider the string $xz = xy^0z$. This string starts with an a-block consisting of less than p many as, followed by one b, followed by an a-block consisting of exactly p many as. Hence, xz is not an element of B. This is a contradiction, because, by the pumping lemma, this string is an element of B. So we have a contradiction, and we can conclude that B is not regular.

Next, we do

$$C = \{a^{2^n} : n \geq 0\}.$$

Assume the language C is regular. Let $p \geq 1$ be the pumping length, as given by the pumping lemma. Let $s = a^{2^p}$. Then $s \in C$ and $|s| = 2^p \geq p$. Hence, by the pumping lemma, we can write $s = xyz$, where

1. $y \neq \epsilon$,
2. $|xy| \leq p$, and
3. $xy^iz \in C$, for all $i \geq 0$.

Since $y \neq \epsilon$ and $|xy| \leq p$, the string y has the form $y = a^k$, for some integer k with $1 \leq k \leq p$. Consider the string

$$xy^2z = xyyz = a^{2^p+k}.$$

The length of this string is equal to $2^p + k$. Since $k \geq 1$, we have $2^p + k > 2^p$. Since $k \leq p$, we have $2^p + k \leq 2^p + p < 2^p + 2^p = 2^{p+1}$. Hence,

$$2^p < |xy^2z| < 2^{p+1},$$

that is, the length of the string xy^2z is strictly between two consecutive powers of 2. But this means that xy^2z is not in the language C. This is a contradiction, because, by the pumping lemma, this string is an element of C. So we have a contradiction, and we can conclude that C is not regular.

Finally, we do

$$D = \{a^n b^m c^\ell : n \geq 0, m \geq 0, \ell \geq 0, n^2 + m^2 = \ell^2\}.$$

Assume the language D is regular. Let $p \geq 1$ be the pumping length, as given by the pumping lemma. Let $s = a^{3p}b^{4p}c^{5p}$. Since $(3p)^2 + (4p)^2 = (5p)^2$, the string s belongs to D. Also, $|s| = 12p \geq p$. Hence, by the pumping lemma, we can write $s = xyz$, where

1. $y \neq \epsilon$,
2. $|xy| \leq p$, and
3. $xy^iz \in D$, for all $i \geq 0$.

Since $y \neq \epsilon$ and $|xy| \leq p$, the string y has the form $y = a^k$, for some integer k with $1 \leq k \leq p$. Consider the string

$$xy^2z = xyyz = a^{3p+k}b^{4p}c^{5p}.$$

Since $k \geq 1$, we have $(3p+k)^2 + (4p)^2 \neq (5p)^2$. Thus, xy^2z is not in the language D. This is a contradiction, because, by the pumping lemma, this string is an element of D. So we have a contradiction, and we can conclude that D is not regular.

Question 5: Consider the language

$$A = \{a^i b^j c^j : i \geq 1, j \geq 0\} \cup \{b^j c^k : j \geq 0, k \geq 0\}.$$

Show that A satisfies the conclusion of the pumping lemma for $p = 1$. Thus, show that every string s in A whose length is at least one can be pumped.

(It can be shown that A is not a regular language. In other words, this example shows that the converse of the pumping lemma does, in general, not hold.)

Solution: The language A is the union of

$$A_1 = \{a^i b^j c^j : i \geq 1, j \geq 0\}$$

and

$$A_2 = \{b^j c^k : j \geq 0, k \geq 0\}.$$

We have to show the following: Every non-empty string s in $A = A_1 \cup A_2$ can be written as $s = xyz$, such that

1. $y \neq \epsilon$,
2. $|xy| \leq 1$, and
3. $xy^\ell z \in A$, for all $\ell \geq 0$.

First, let s be an arbitrary non-empty string in A_1. Thus, $s = a^i b^j c^j$ for some integers $i \geq 1$ and $j \geq 0$. Define $x = \epsilon$, $y = a$, and $z = a^{i-1} b^j c^j$. Then obviously $s = xyz$. Moreover,

1. $y \neq \epsilon$,
2. $|xy| = 1 \leq 1$, and
3. for any $\ell \geq 0$,

$$xy^\ell z = \begin{cases} b^j c^j \in A_2 \subseteq A & \text{if } \ell = 0 \text{ and } i = 1, \\ a^{\ell+i-1} b^j c^j \in A_1 \subseteq A & \text{if } \ell \geq 1 \text{ or } i \geq 2. \end{cases}$$

Next, let s be an arbitrary non-empty string in A_2. Thus, $s = b^j c^k$ for some integers $j \geq 0$ and $k \geq 0$. Since $s \neq \epsilon$, we have $j + k \geq 1$. Let $x = \epsilon$, let y be the string of length one whose only symbol is the first symbol of s, and let z be the remaining part of s. Then obviously $s = xyz$. Moreover,

1. $y \neq \epsilon$,
2. $|xy| = 1 \leq 1$, and
3. for any $\ell \geq 0$,

$$xy^\ell z = \begin{cases} b^{\ell+j-1} c^k \in A_2 \subseteq A & \text{if } j \geq 1 \text{ and } k \geq 0, \\ c^{\ell+k-1} \in A_2 \subseteq A & \text{if } j = 0 \text{ and } k \geq 1. \end{cases}$$